ISRAEL JOURNAL OF MATHEMATICS **137** (2003), 183–188

ON SUBGROUPS CONTAINING NON-TRIVIAL NORMAL SUBGROUPS

BY

MARCEL HERZOG

School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences
Tel Aviv University, Tel Aviv 69978, Israel
e-mail: herzogm@post.tau.ac.il

AND

GIL KAPLAN

Department of Computer Sciences, The Academic College of Tel Aviv–Yaffo
4 Antokolsky St., Tel Aviv 64044, Israel
e-mail: gil6k@netvision.net.il

AND

ANDREA LUCCHINI

Dipartimento di Elettronica per l'Automazione, Univesità degli Studi di Brescia
Via Branze, 25123 Brescia, Italy
e-mail: andrea.lucchini@unibs.it

ABSTRACT

We prove that if $A \neq 1$ is a subgroup of a finite group G and the order of an element in the centralizer of A in G is strictly larger (larger or equal) than the index $[G : A]$, then A contains a non-trivial characteristic (normal) subgroup of G. Consequently, if A is a stabilizer in a transitive permutation group of degree $m > 1$, then $exp(Z(A)) < m$. These theorems generalize some recent results of Isaacs and the authors.

Received July 22, 2002 and in revised form November 12, 2002

I. Introduction

All groups in this paper are finite. For a group G we denote the center, the Frattini subgroup and the exponent of G by $Z(G)$, $\Phi(G)$ and $exp(G)$, respectively. The notation K *char* G stands for "K is a characteristic subgroup of G". Moreover, we denote by $Exp(G)$ the maximal order of a cyclic subgroup of G. We have $exp(Z(G)) \leq Exp(G) \leq exp(G)$ for any group G, and $Exp(G) = exp(G)$ if G is nilpotent.

In [3] Isaacs proved the following theorem, which we state in a re-formulated form:

THEOREM I: *Let $G \neq 1$ be a group with an abelian subgroup A such that $exp(A) \geq [G : A]$. Then A contains a non-trivial normal subgroup of G.*

This theorem generalizes the following result of Lucchini [4], which was also proved independently and using different methods by the other authors of this paper in [1]:

THEOREM L: *Let $G \neq 1$ be a group with a cyclic subgroup A such that $|A| \geq |G|^{1/2}$. Then A contains a non-trivial normal subgroup of G.*

In [1] a related result was also proved:

THEOREM HK: *Let G be a group with a cyclic subgroup A such that $|A| > |G|^{1/2}$. Then A contains a non-trivial characteristic subgroup of G.*

In this paper we generalize all these results in two directions. Firstly, we allow A to be an arbitrary subgroup of G instead of being cyclic or abelian, and secondly, we assume only that $Exp(C_G(A))|A| \geq |G|$ (or $>$), which is weaker than $exp(A)|A| \geq |G|$ (or $>$) in the case when A is abelian. We prove

THEOREM A: *Let A be a subgroup of a group G satisfying $Exp(C_G(A)) > [G : A]$. Then A contains a non-trivial characteristic subgroup of G.*

We also prove

THEOREM B: *Let $A \neq 1$ be a subgroup of a group G satisfying $Exp(C_G(A)) \geq [G : A]$. Then A contains a non-trivial normal subgroup of G.*

In the language of permutation group theory, which is used in the papers of Isaacs and Lucchini, Theorem B implies

THEOREM C: *Suppose that $A \neq 1$ is a point stabilizer in a transitive permutation group G of degree $m > 1$. Then $Exp(C_G(A)) < m$.*

Another application of Theorem B is the following theorem, where orders of certain non-trivial elements are compared with sizes of their conjugacy classes.

THEOREM D: *If $1 \neq g \in G$ and $C_G(g)$ contains no non-trivial normal subgroup of G, then*

$$o(g) < [G : C_G(g)].$$

If A is abelian, then, as mentioned above, $exp(A) \leq Exp(C_G(A))$ and hence our numerical assumptions in Theorem B are weaker than those of Theorem I.

Since $Exp(C_G(A)) \geq exp(Z(A))$ for any subgroup A of G, our theorems A–C yield the following corollaries:

COROLLARY A1: *Let A be a subgroup of a group G satisfying $exp(Z(A)) > [G : A]$. Then A contains a non-trivial characteristic subgroup of G.*

COROLLARY B1: *Let $A \neq 1$ be a subgroup of a group G satisfying $exp(Z(A)) \geq [G : A]$. Then A contains a non-trivial normal subgroup of G.*

COROLLARY C1: *Suppose that A is a point stabilizer in a transitive permutation group G of degree $m > 1$. Then $exp(Z(A)) < m$.*

We also mention the following immediate corollary of Theorem A:

COROLLARY A2: *Let G be a group with an abelian subgroup A such that $exp(A) > [G : A]$. Then A contains a non-trivial characteristic subgroup of G.*

II. Proofs

In our proofs we shall use Lemma 1.1 in [4], which is restated below:

LEMMA L1: *Let G be a finite group with the property that all its non-trivial normal subgroups are non-abelian. Then*

(a) $|G| \geq |H||C_G(H)|$ *for each subgroup H of G,*
(b) $|G| > |H||C_G(H)|$ *for each non-trivial solvable subgroup H of G.*

We start with

Proof of Theorem A: We apply induction on $|G|$. Suppose, first, that there exists a non-trivial characteristic subgroup K of G, such that $AK \neq G$, and choose M to be maximal with respect to this property. Let B be a cyclic subgroup of $C_G(A)$ of maximal order, which, by our assumptions, satisfies $|B| = Exp(C_G(A)) > [G : A]$. We claim that $|B \cap M| > [M : A \cap M]$. Indeed, suppose on the contrary that $|B \cap M| \leq [M : A \cap M] = [AM : A]$. Then

$$|BM/M| = |B|/|B \cap M| > [G : A]/[AM : A] = [G/M : AM/M]$$

and hence AM/M is a subgroup of G/M such that $Exp(C_{G/M}(AM/M)) > [G/M : AM/M]$. It follows by the induction hypothesis applied to G/M that AM/M contains a non-trivial characteristic subgroup of G/M, say L/M. Since M *char* G, we have also L *char* G, and since $L > M$ and $AL = AM \neq G$, we have obtained a contradiction to the maximality of M. Thus our claim that $|B \cap M| > [M : A \cap M]$ has been proved. Consequently, $A \cap M$ is a subgroup of M such that $Exp(C_M(A \cap M)) > [M : A \cap M]$, and it follows by the induction hypothesis applied to M that there exists a non-trivial subgroup $D \leq A \cap M$, which is characteristic in M. But M *char* G, so D *char* G, as required.

So assume, from now on, that $1 \neq K$ *char* G implies $AK = G$. Notice that A is a subgroup of G satisfying

$$|A||C_G(A)| \geq |A||B| > |A|[G : A] = |G|,$$

and by Lemma L1(a) we may conclude that G has a non-trivial abelian normal subgroup. It follows that G has a minimal non-trivial abelian characteristic p-subgroup, say P, for some prime p. By our assumption $AP = G$ and so $[G : A]$ is a power of p.

Suppose that $Z(G) = 1$. Let F denote the field of p elements. Then P is a simple $FAut(G)$-module and it follows by Clifford's theorem (Theorem 2.14 in [2]) that P is a semisimple $FInn(G)$-module. Consequently, as $A \cap P \triangleleft G$, there exists $X \leq P$ such that $X \triangleleft G$ and $P = (A \cap P) \times X$. Hence $|X| = [G : A]$ and, since $Z(G) = 1$, the group B acts faithfully on X by conjugation. Thus, by Corollary 2 in [1], we have $Exp(C_G(A)) = |B| < |X| = [G : A]$, in contradiction to our assumptions.

So $Z(G) \neq 1$ and by our assumptions $AZ(G) = G$. Hence $G' \leq A$ and the theorem holds if $G' \neq 1$. It remains to deal with the case that G is abelian. If A is not a p-group, then it clearly contains a non-trivial characteristic subgroup of G. So we may assume that A, and hence G, is an abelian p-group. If $\Phi(G) \neq 1$, then $A\Phi(G) = G$, which implies that $G = A$ and the theorem holds. So we may assume that G is an elementary abelian p-group. Since $p = Exp(C_G(A)) > [G : A]$, we conclude again that $A = G$, and the theorem holds in all cases. ∎

The proof of Theorem B is similar to that of Theorem A.

Proof of Theorem B: We apply induction on $|G|$. Suppose, first, that there exists a non-trivial normal subgroup N of G such that $AN \neq G$, and choose M to be maximal with respect to this property. Let B be a cyclic subgroup of $C_G(A)$ of maximal order, which satisfies $|B| \geq [G : A]$. If $|B \cap M| > [M : A \cap M]$, then applying Theorem A to M and $A \cap M$, we conclude that there exists a non-trivial subgroup $D \leq A \cap M$ such that D is characteristic in M and hence normal in G, as required. So suppose that $|B \cap M| \leq [M : A \cap M]$. Then it follows, by arguments used in the proof of Theorem A, that $|BM/M| \geq [G/M : AM/M]$. If $A \nleq M$ holds, then by induction AM/M contains a non-trivial normal subgroup of G/M, say L/M. Since $AL = AM \neq G$, we get a contradiction to the maximality of M. If $A \leq M$, then $|BM/M| \geq |G/M|$ and hence $G = BM$. Since $|B| \geq [G : A]$, it follows that $|B \cap M| \geq [M : A]$ and, applying induction to M and A, we may conclude that A contains a non-trivial normal subgroup D of M. But $G = BM$, so $D \triangleleft G$, as required.

So assume from now on that $1 \neq K \triangleleft G$ implies $AK = G$. By our assumptions

$$|A||C_G(A)| \geq |A||B| \geq |A|[G : A] = |G|.$$

If $|A||C_G(A)| = |G|$, then $C_G(A) = B$ is cyclic and

$$|B||C_G(B)| \geq |B||A| = G,$$

which implies by Lemma L1(b) that G has a non-trivial abelian normal subgroup. The same conclusion may be drawn in the complementary case, i.e., $|A||C_G(A)| > |G|$, by means of Lemma L1(a). It follows that G has a minimal non-trivial abelian characteristic p-subgroup, say P, for some prime p. By our assumptions $G = AP$, and hence $[G : A]$ is a power of p.

If $Z(G) = 1$ then, as in the proof of Theorem A, there exists a complement X of $A \cap P$ in P, which is normal in G, and $Exp(C_G(A)) = |B| < |X| = [G : A]$, in contradiction to our assumptions. Thus $Z(G) \neq 1$ and $G = AZ(G)$. Hence $1 \neq A \trianglelefteq G$, as required, and the proof is complete. ∎

We conclude with

Proof of Theorem D: Denote $A = C_G(g)$ and suppose, on the contrary, that $o(g) \geq [G : A]$. Clearly $g \in C_G(A)$ and hence $Exp(C_G(A)) \geq o(g) \geq [G : A]$. This implies, by Theorem B, that $A = C_G(g)$ contains a non-trivial normal subgroup of G, contradicting our assumptions. ∎

References

[1] M. Herzog and G. Kaplan, *Large cyclic subgroups contain non-trivial normal subgroups*, J. Group Theory **4** (2001), 247–253.

[2] B. Huppert, *Character Theory of Finite Groups*, De Gruyter Expositions in Mathematics 25, Walter de Gruyter, Berlin, 1998.

[3] I. M. Isaacs, *Abelian point stabilizers in transitive permutation groups*, Proceedings of the American Mathematical Society **130** (2002), 1923–1925.

[4] A.Lucchini, *On the order of transitive permutation groups with cyclic point-stabilizer*, Rendiconti di Matematica dell'Accademia delle Lincei **9** (1998), 241–243.